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A boundary integral equation method for a Neumann boundary problem for force-free fields

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SUMMARY

A Neumann boundary value problem for the equation rot $v - \lambda v = 0$ is considered in $\mathbb{R}^3$ and $\mathbb{R}^2$. The approach is by transforming the boundary value problem into an equivalent boundary integral equation deduced from a representation formula for solutions of rot $v - \lambda v = 0$ based on the fundamental solution of the Helmholtz equation. In particular, for the two-dimensional case a detailed discussion of the integral equation is carried out including the approximate solution by numerical integration.

1. Introduction

Force-free fields in $\mathbb{R}^3$ are solutions of one of the two equivalent equations

$$[\text{rot}\, v, v] = 0 \tag{1.1}$$

or

$$\text{rot}\, v - \lambda v = 0 \tag{1.2}$$

where λ denotes a scalar which, in general, is space dependent. They may be regarded as static magnetic fields for which the Lorentz force vanishes and they describe the equilibrium of an electrically conducting liquid – for instance a plasma – in the presence of a magnetic field ([3], pp. 188, [4], pp. 35, [14]). In hydrodynamics solutions of (1.1) or (1.2) are also called Beltrami fields ([1], [18], p. 43, [19], p. 188, [20], p. 68, 76) and they correspond to steady incompressible rotational fluid flows which possess Bernoulli constants independent of the streamlines.

In several previous papers [8], [9], [10] the author considered the treatment of a Neumann boundary value problem by integral equation methods using two different approaches. The first method, described in [8] and [9], is based on Green's matrices for appropriate boundary value problems of potential theory and results in a volume integral equation for the unknown field which is equivalent to the boundary value problem. The second method, described in [10], is based on a representation theorem for inhomogeneous harmonic vector fields and leads to a system of one volume and one surface integral equation for the volume and surface vortices of the unknown field which again is equivalent to the boundary value problem. Comparing the two approaches, the first one gives results on the existence of eigenvalues λ because of selfadjointness properties of the volume integral operator which are not obtainable by the second method. On the other hand, the second approach enables one to get results on the existence of solutions to the boundary value problem in terms of the familiar Fredholm alternative which cannot be found by the first method. Both approaches can be used for λ not necessarily constant.

In this paper we shall develop a third approach which makes use of the property that any force-free field with constant λ satisfies the vector Helmholtz equation

$$\Delta v + \lambda^2 v = 0.$$

In the first part of the paper we shall give a representation theorem for solutions of (1.2) with constant λ based on the fundamental solution to the Helmholtz equation. Then this representation theorem is used to obtain a boundary integral equation equivalent to the boundary value problem. In the second part this integral equation will be discussed in the two dimensional case of force-free fields in an infinite cylinder. In particular we shall outline a method for the numerical solution of the integral equation and provide a numerical example.

The main advantage of this new approach lies in the fact that the integral equation contains only boundary integral operators which is of considerable importance for numerical approximations. As compared with the two previous methods it is confined to constant λ.

2. A representation theorem

Let B be a bounded domain in $\mathbb{R}^3$. The boundary of B, denoted by S, is assumed to be connected and to belong to the class C^2. The complement of B is designated by $\hat{B} := \mathbb{R}^3 \backslash \overline{B}$. By n we denote the unit normal to S directed into $\hat{B}$.

Let λ be a real number and denote by

$$\Phi(x,y) := \frac{1}{4\pi} \frac{e^{i\lambda|x-y|}}{|x-y|} \tag{2.1}$$

the fundamental solution to the Helmholtz equation in three dimensions.

We present the following representation theorem:

Theorem 2.1. *Let $v \in C^1(B) \cap C(\overline{B})$ be a complex valued vectorfield such that* div v, rot $v \in C(\overline{B})$. *Then there holds**

$$v = -\operatorname{grad} U + \operatorname{rot} A + \lambda A \quad \text{in } B \tag{2.2}$$

where

* By (a,b), $[a,b]$ and (a,b,c) we denote the scalar product, vector product and triple product of the vectors a,b,c, respectively

$$U(x) := \int_B \Phi(x,y)\operatorname{div} v(y)dy - \int_S \Phi(x,y)\,(n(y),v(y))ds(y), \tag{2.3}$$

$$A(x) := \int_B \Phi(x,y)\,\{\operatorname{rot} v(y) - \lambda v(y)\}dy - \int_S \Phi(x,y)\,[n(y),v(y)]ds(y). \tag{2.4}$$

Furthermore

$$\operatorname{div} A + \lambda U = 0 \quad \textit{in } B. \tag{2.5}$$

Proof: We choose an arbitrary fixed point $x \in B$ and circumscribe it with a sphere $K_\rho(x) := \{y \in \mathbb{R}^3 \mid |x - y| = \rho\}$. We assume the radius small enough so that $K_\rho(x) \subset B$ and direct the unit normal n to $K_\rho(x)$ into the interior of $K_\rho(x)$. For all $y \in B_\rho := \{y \in B \mid |x - y| > \rho\}$ there holds

$$\operatorname{div}_x \Phi v(y) = \Phi \operatorname{div}_y v(y) - \operatorname{div}_y \Phi v(y), \tag{2.6}$$

$$\operatorname{rot}_x \Phi v(y) - \lambda \Phi v(y) = \Phi\{\operatorname{rot}_y v(y) - \lambda v(y)\} - \operatorname{rot}_y \Phi v(y). \tag{2.7}$$

Subtracting the gradient of (2.6) from the rotation of (2.7) we obtain

$$- \operatorname{grad}_x[\Phi \operatorname{div}_y v(y) - \operatorname{div}_y \Phi v(y)] + \operatorname{rot}_x[\Phi\{\operatorname{rot}_y v(y) - \lambda v(y)\} - \operatorname{rot}_y \Phi v(y)]$$

$$= - \operatorname{grad} \operatorname{div}_x \Phi v(y) + \operatorname{rot} \operatorname{rot}_x \Phi v(y) - \lambda \operatorname{rot}_x \Phi v(y)$$

$$= - \Delta_x \Phi v(y) - \lambda \operatorname{rot}_x \Phi v(y)$$

$$= - \lambda[\Phi\{\operatorname{rot}_y v(y) - \lambda v(y)\} - \operatorname{rot}_y \Phi v(y)].$$

Now we integrate over B_ρ and apply Gauss' theorem to obtain

$$- \operatorname{grad}_x \left[\int_{B_\rho} \Phi \operatorname{div}_y v(y)dy - \int_S \Phi(n(y),v(y))ds(y) \right]$$

$$+ \operatorname{rot}_x \left[\int_{B_\rho} \Phi\{\operatorname{rot}_y v(y) - \lambda v(y)\}dy - \int_S \Phi[n(y),v(y)]ds(y) \right]$$

$$+ \lambda \left[\int_{B_\rho} \Phi\{\operatorname{rot}_y v(y) - \lambda v(y)\}dy - \int_S \Phi[n(y),v(y)]ds(y) \right]$$

$$= - \operatorname{grad}_x \int_{K_\rho} \Phi[n(y),v(y)]ds(y) + \operatorname{rot}_x \int_{K_\rho} \Phi[n(y),v(y)]ds(y)$$

$$+ \lambda \int_{K_\rho} \Phi[n(y),v(y)]ds(y).$$

Since on K_ρ there holds

$$\Phi(x,y) = \frac{e^{i\lambda\rho}}{4\pi\rho}\;, \quad \operatorname{grad}_y\Phi(x,y) = \left(\frac{1}{\rho} - i\lambda\right)\frac{e^{i\lambda\rho}}{4\pi\rho}\; n(y),$$

a straightforward calculation shows that the right hand side of the previous equation tends to $\nu(x)$ as $\rho \to 0$, whence (2.2) follows.

Adding the divergence of (2.7) to equation (2.6) multiplied by λ we get

$$0 = \operatorname{div}_x[\Phi\{\operatorname{rot}_y\nu(y) - \lambda\nu(y)\} - \operatorname{rot}_y\Phi\nu(y)] + \lambda[\Phi\,\operatorname{div}_y\nu(y) - \operatorname{div}_y\Phi\nu(y)].$$

Again we integrate over B_ρ and apply Gauss' theorem to obtain

$$\operatorname{div}_x\left[\int_{B_\rho}\Phi\{\operatorname{rot}_y\nu(y) - \lambda\nu(y)\}dy - \int_S \Phi[n(y),\nu(y)]ds(y)\right]$$

$$+\lambda\left[\int_{B_\rho}\Phi\,\operatorname{div}_y\nu(y)dy - \int_S \Phi(n(y),\nu(y))ds(y)\right]$$

$$= \operatorname{div}_x\int_{K_\rho}\Phi[n(y),\nu(y)]ds(y) + \lambda\int_{K_\rho}\Phi(n(y),\nu(y))ds(y).$$

By straightforward calculation it is verified that the right hand side tends to zero as $\rho \to 0$, whence (2.5) follows.

The representation theorem 2.1 generalizes Cauchy's integral formula for vector fields ([16], p. 97) which may be regarded as the special case $\lambda = 0$ of the case of arbitrary λ.

3. Boundary value problem and equivalent integral equation

We shall consider the following boundary value problem from the theory of force-free fields.

Problem $K(B)$: *Given a real number* $\lambda \neq 0$, *a vector field* $u \in C^{1,\alpha}(\overline{B})$ *and a function* $\epsilon \in C^{0,\alpha}(S)$, $0 < \alpha < 1$, *find a vector field* $\nu \in C^1(B) \cap C(\overline{B})$ *satisfying the differential equation*

$$\operatorname{rot}\nu - \lambda\nu = u \quad \text{in } B \tag{3.1}$$

and assuming normal components

$$-(n,\nu) = \epsilon \quad \text{on } S. \tag{3.2}$$

By Stokes' theorem we readily observe that the condition

$$\int_S \{(n,u) - \lambda\epsilon\} ds = 0 \tag{3.3}$$

is necessary for the existence of a solution to problem $K(B)$. In the subsequent analysis we shall assume that this solvability condition is satisfied.

We derive a boundary integral equation for the tangential components of a solution to problem $K(B)$.

Theorem 3.1. *Let v be a solution of the boundary value problem $K(B)$. Then the tangential component*

$$\gamma := -[n,v] \quad \text{on } S \tag{3.4}$$

solves the boundary integral equation

$$\tfrac{1}{2}\gamma(x) + \int_S [n(x), \operatorname{rot}_x \Phi\gamma(y) + \lambda\Phi\gamma(y)] ds(y) \tag{3.5}$$

$$= \Bigg[n(x), \operatorname{grad}_x \left\{ \int_B -\frac{1}{\lambda}\,\Phi \operatorname{div} u(y) dy + \int_S \Phi\epsilon(y) ds(y) \right\}$$

$$- \operatorname{rot}_x \int_B \Phi u(y) dy - \lambda \int_B \Phi u(y) dy \Bigg], \quad x \in S.$$

Proof: From the differential equation (3.1) we find

$$\operatorname{div} v = -\frac{1}{\lambda} \operatorname{div} u \quad \text{in } B. \tag{3.6}$$

Therefore, by the representation theorem 2.1 we can write

$$v(x) = -\operatorname{grad} \left\{ \int_B -\frac{1}{\lambda}\,\Phi \operatorname{div} u(y) dy + \int_S \Phi\epsilon(y) ds(y) \right\}$$

$$+ \operatorname{rot} \left\{ \int_B \Phi u(y) dy + \int_S \Phi\gamma(y) ds(y) \right\}$$

$$+ \lambda \left\{ \int_B \Phi u(y) dy + \int_S \Phi\gamma(y) ds(y) \right\}, \quad x \in B.$$

From this, with the help of the jump relations for surface potentials ([5], [17], p. 194) the integral equation (3.5) is obtained by letting x tend to the boundary S. We point out that for the application of the jump relations it suffices to have γ continuous on S, but ϵ has to be uniformly Hölder continuous.

In order to state the converse of Theorem 3.1 we give the

Definition 3.2. *The number λ is called regular with respect to the boundary value problem $K(B)$ if for all solutions $w \in C^1(B) \cap C(\overline{B}), a \in C^2(B) \cap C(\overline{B})$ of the system of differential equations*

$$\operatorname{rot} w - \lambda w = u + \operatorname{grad} a \qquad (3.7)$$

in B

$$\Delta a + \lambda^2 a = 0 \qquad (3.8)$$

satisfying the boundary conditions

$$-(n,w) = \epsilon \qquad (3.9)$$

on S

$$a = 0 \qquad (3.10)$$

there follows $a = 0$ in B.

Obviously, the set

$$\Lambda(B) := \{\lambda \neq 0 \mid \lambda \text{ is not regular with respect to } K(B)\}$$

is a subset of the countable set of interior Dirichlet eigenvalues λ for which (3.8) and (3.10) have a nontrivial solution a for which

$$\int_B a dx = 0. \qquad (3.11)$$

In Sec. 4 we shall provide an example for a nonempty $\Lambda(B)$.

In order to decide whether λ is regular or not is has to be checked whether the additional uniqueness property required in Definition 3.2 is satisfied or not. This can be done either by establishing beforehand that λ is not an eigenvalue of the Dirichlet problem or just a simple eigenvalue with $\int_B a dx \neq 0$ for the eigenfunction a, or it will automatically manifest itself during the numerical calculation. Since in the case of a non-regular λ, as we shall see later, the integral equation (3.5) has additional solutions, the linear system obtained from the integral equation by numerical integration will become ill-conditioned.

Theorem 3.3. *Let λ be regular with respect to $K(B)$ and let γ be a continuous solution of the integral equation* (3.5). *Define*

$$U(x) := -\frac{1}{\lambda} \int_B \Phi \operatorname{div} u(y) dy + \int_S \Phi \epsilon(y) ds(y), \qquad (3.12)$$

$$A(x) := \int_B \Phi u(y) dy + \int_S \Phi \gamma(\gamma) ds(y). \qquad (3.13)$$

Then

$$v := -\operatorname{grad} U + \operatorname{rot} A + \lambda A \tag{3.14}$$

is a solution to the boundary value problem $K(B)$.

Proof: Since $u \in C^{1,\alpha}(\bar{B})$, by virtue of the regularity properties of volume potentials ([5], [12], p. 27) we have $U,A \in C^2(B)$ and $U,A \in C^2(\hat{B})$ and

$$\Delta U + \lambda^2 U = \begin{cases} \frac{1}{\lambda} \operatorname{div} u & \text{in } B, \\ 0 & \text{in } \hat{B}; \end{cases} \tag{3.15}$$

$$\Delta A + \lambda^2 A = \begin{cases} -u & \text{in } B, \\ 0 & \text{in } \hat{B}. \end{cases} \tag{3.16}$$

Using (3.15) and (3.16), we find $v \in C^1(B)$ and $v \in C^1(\hat{B})$ and

$$\operatorname{rot} v - \lambda v = \begin{cases} \operatorname{grad}(\operatorname{div} A + \lambda v) + u & \text{in } B, \\ \operatorname{grad}(\operatorname{div} A + \lambda v) & \text{in } \hat{B}. \end{cases} \tag{3.17}$$

Next we observe that the right hand side of the integral equation (3.5) is of class $C^{0,\alpha}(S)$ because $u \in C^{1,\alpha}(\bar{B})$ and $\epsilon \in C^{0,\alpha}(S)$. Since the integral operator on the left hand side of (3.5) maps $C(S)$ into $C^{0,\alpha}(S)$ ([5], p. 62, [6], Theorem 5.1, [21], Lemma 6) any continuous solution γ automatically is of class $C^{0,\alpha}(S)$. Therefore, due to the regularity properties of surface potentials with uniformly Hölder continuous densities, v can be continued uniformly Hölder continuously into S from both sides. If we distinguish the limits obtained by approaching S from inside $\hat{B}$ and B by the indices + and −, respectively, the jump relations for the surface potentials lead to

$$v_+ - v_- = \epsilon n + [\gamma, n] \quad \text{on } S, \tag{3.18}$$

and, with the aid of the integral equation (3.5),

$$[n, v_+] = 0 \quad \text{on } S. \tag{3.19}$$

Since div A in $\hat{B}$ is a solution to the Helmholtz equation satisfying the Sommerfeld radiation condition by Green's representation theorem we can write

$$\operatorname{div} A(x) = \int_{\tilde{S}} \left\{ \operatorname{div} A(y) \frac{\partial \Phi}{\partial n(y)} - \Phi \frac{\partial}{\partial n(y)} \operatorname{div} A(y) \right\} ds(y), \quad x \in \hat{B},$$

where $\widetilde{S}$ denotes a surface parallel to S separating the point x from S. Using (3.17) and Stokes' theorem we transform

$$\int_{\widetilde{S}} \Phi \frac{\partial}{\partial n} \operatorname{div} A \, ds = \int_{\widetilde{S}} \Phi (n, \operatorname{rot} v - \lambda v - \lambda \operatorname{grad} U) ds$$

$$= \int_{\widetilde{S}} [(n, v, \operatorname{grad} \Phi) - \lambda \Phi (n, v + \operatorname{grad} U)] ds.$$

Now we are able to pass to the limit $\widetilde{S} \to S$ and with the aid of (3.19) we obtain

$$\operatorname{div} A(x) = \int_S \left\{ \operatorname{div} A(y) \frac{\partial \Phi}{\partial n(y)} + \mu(y) \Phi \right\} ds(y), \quad x \in \hat{B}, \tag{3.20}$$

where we have set $\mu := \lambda (n, v_+ + \operatorname{grad} U_+)|_S$. Since $\operatorname{div} A|_S \in C^{0,\alpha}(S)$ and $\mu \in C^{0,\alpha}(S)$ we now may let x tend to the boundary and find the integral equation

$$\tfrac{1}{2} \operatorname{div} A(x) - \int_S \operatorname{div} A(y) \frac{\partial \Phi}{\partial n(y)} ds(y) = \int_S \mu(y) \Phi ds(y), \quad x \in S. \tag{3.21}$$

The right hand side of (3.21) belongs to $C^{1,\alpha}(S)$ since for the density we have $\mu \in C^{0,\alpha}(S)$. Thus, because the integral operator in equation (3.21) maps $C^{0,\alpha}(S)$ into $C^{1,\alpha}(S)$ ([12], p. 42, [21], Lemma 7) we conclude $\operatorname{div} A|_S \in C^{1,\alpha}(S)$. But then finally from (3.20) we see $\operatorname{div} A \in C^{1,\alpha}(\bar{\hat{B}})$ because double layer potentials with densities of class $C^{1,\alpha}(S)$ belong to $C^{1,\alpha}(\bar{\hat{B}})$ ([12], p. 40, [21], Lemma 4).

From the transformation

$$\operatorname{div} A(x) = \int_B \Phi \operatorname{div} u(y) dy - \int_S \Phi(n(y), u(y)) ds(y) + \operatorname{div} \int_S \Phi \gamma(y) ds(y)$$

we deduce $\operatorname{div} A \in C^2(B) \cap C^{0,\alpha}(\bar{B})$. Define

$$a := \operatorname{div} A + \lambda U. \tag{3.22}$$

Then, using (3.15) and (3.16), we find

$$\Delta a + \lambda^2 a = 0 \quad \text{in } B \text{ and } \hat{B} \tag{3.23}$$

and with the help of the jump relations we find

$$a_+ = a_- \quad \text{on } S. \tag{3.24}$$

Define

$$w := \lambda v + \operatorname{grad} a \quad \text{in } \hat{B}. \tag{3.25}$$

From (3.17) we deduce

$$\operatorname{rot} w - \lambda w = 0 \quad \text{in } \hat{B} \tag{3.26}$$

and

$$\Delta w + \lambda^2 w = 0 \quad \text{in } \hat{B}. \tag{3.27}$$

By straightforward calculation it can be shown that the radiation condition

$$\begin{aligned} &w(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \\ &\left[\operatorname{rot} w(x), \frac{x}{|x|}\right] - i\lambda w(x) = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty, \end{aligned} \tag{3.28}$$

holds uniformly with respect to all directions $x/|x|$. Let $K_R := \{x \in \mathbb{R}^3 \mid |x| = R\}$ and assume the radius R of this sphere large enough such that $K_R \subset \hat{B}$. Let n denote the outward drawn unit normal to K_R and let $\hat{B}_R := \{x \in \hat{B} \mid |x| < R\}$. Then, using Gauss' and Stokes' theorem, from (3.26) and (3.19) we derive

$$\int_{K_R} (n, \bar{w}, \operatorname{rot} w)\, ds = \lambda \int_{\hat{B}_R} \operatorname{div}[\bar{w}, w]\, dx + \lambda \int_S (n, \bar{w}, w)\, ds$$

$$= \lambda \int_S (n, \operatorname{grad} \bar{a}, \operatorname{grad} a)\, ds = 0.$$

On the other hand, from (3.28) there follows

$$\int_{K_R} (n, \bar{w}, \operatorname{rot} w)\, ds = i\lambda \int_{K_R} |w|^2 ds + o(1), \quad R \to \infty.$$

Hence,

$$\int_{K_R} |w|^2 ds = o(1), \quad R \to \infty$$

and from this and (3.27) we conclude

$$w = 0 \quad \text{in } \hat{B}$$

by Rellich's Lemma ([12], p. 161).

We now have

$$\lambda v + \operatorname{grad} a = 0 \quad \text{in } \hat{B} \tag{3.29}$$

and from (3.19) we deduce

$$a = a_0 = \text{const} \quad \text{on } S. \tag{3.30}$$

Therefore, from the regularity properties of solutions to the Dirichlet problem for the Helmholtz equation ([12], p. 157) we obtain $a \in C^{1,\alpha}(\overline{B})$. Using Green's and Stokes' theorem with the aid of (3.23), (3.29), (3.30), (3.18), (3.17) and (3.3) we get

$$Im \left\{ \int_{K_R} \bar{a} \frac{\partial a}{\partial n} ds \right\} = Im \left\{ \int_{\hat{B}_R} [\bar{a}\Delta a + |\operatorname{grad} a|^2] dx + \int_S \bar{a} \frac{\partial a}{\partial n} ds \right\}$$

$$= -\lambda \, Im \left\{ \bar{a}_0 \int_S [(n,v_-) + \epsilon] ds \right\}$$

$$= Im \left\{ \bar{a}_0 \int_S \left[-(n, \operatorname{rot} v_-) + (n,u) - \lambda\epsilon + \frac{\partial a_-}{\partial n} \right] ds \right\}$$

$$= Im \left\{ \int_B [\bar{a}\Delta a + |\operatorname{grad} a|^2] dx \right\} = 0.$$

On the other hand, since obviously the radiation condition

$$a(x) = O\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

$$\left(\operatorname{grad} a(x), \frac{x}{|x|}\right) - i\lambda a(x) = o\left(\frac{1}{|x|}\right), \quad |x| \to \infty,$$

is satisfied uniformly with respect to all directions, we get

$$\int_{K_R} \bar{a} \frac{\partial a}{\partial n} ds = i\lambda \int_{K_R} |a|^2 ds + o(1), \quad R \to \infty.$$

Hence,

$$\int_{K_R} |a|^2 ds = o(1), \quad R \to \infty$$

and again by Rellich's Lemma we get

$$a = 0 \quad \text{in } \hat{B} \tag{3.31}$$

whence

$$v = 0 \quad \text{in } \hat{B} \tag{3.32}$$

follows.

Summarizing our results we have now arrived at

$$\operatorname{rot} v - \lambda v = u + \operatorname{grad} a \quad \text{in } B, \tag{3.33}$$

$$-(n,v) = \epsilon \quad \text{on } S, \tag{3.34}$$

and

$$\Delta a + \lambda^2 a = 0 \quad \text{in } B, \tag{3.35}$$

$$a = 0 \quad \text{on } S. \tag{3.36}$$

Thus, since λ is assumed to be regular with respect to $K(B)$ we obtain $a = 0$ in B and the proof is complete.

When λ is not regular with respect to $K(B)$ we cannot expect the converse of Theorem 3.1 to be valid. In this case, for the additional term in the representation theorem stemming from the extra term grad a on the right hand side of the differential equation (3.7), we calculate

$$-\operatorname{grad} \int_B -\frac{1}{\lambda}\, \Phi \Delta a(y) dy + \operatorname{rot} \int_B \Phi \operatorname{grad} a(y) dy + \lambda \int_B \Phi \operatorname{grad} a(y) dy \tag{3.37}$$

$$= \int_B \{\lambda a(y) \operatorname{grad}_y \Phi - [\operatorname{grad}_y \Phi, \operatorname{grad} a(y)] + \lambda \Phi \operatorname{grad} a(y)\} dy$$

$$= \int_B \{\lambda \operatorname{grad}_y \Phi a(y) - \operatorname{rot}_y(\Phi \operatorname{grad} a(y))\} dy$$

$$= \int_S \{\lambda \Phi a(y) n(y) - \Phi[n(y), \operatorname{grad} a(y)]\} ds(y) = 0.$$

Therefore, by going for the tangential component of the solution w of (3.7) and (3.9) through the same argument as carried out in the proof of Theorem 3.1, we see that the tangential component of w solves the same equation as for the solution v to (3.1) and (3.2). But the tangential components of the two solutions must be different, because otherwise from our calculation (3.37) and the representation theorem we would obtain the contradiction $w = v$.

These results may serve as an example for a boundary integral equation derived from a representation theorem which is not fully equivalent to the boundary value problem.

4. Two-dimensional case

In the subsequent analysis we shall study the two-dimensional case for which, as we immediately shall see, a complete discussion of the boundary integral equation is possible.

To be more precise, we shall consider the boundary value problem $K(B)$ for a vectorfield $v = (v_1(x_1,x_2), v_2(x_1,x_2), v_3(x_1,x_2))$ in a cylinder $B \times \mathbb{R} \subset \mathbb{R}^3$ where B denotes a bounded domain in $\mathbb{R}^2$. Then our previous results remain valid after replacing (2.1) by the fundamental solution

$$\Phi(x,y) = \frac{i}{4} H_0^1(\lambda \,|\, x - y \,|) \tag{4.1}$$

to the Helmholtz equation in two dimensions. Here H_0^1 denotes the Hankel function of the first kind and of order zero. The integrals being understood to be carried out over the cross section B of the cylinder and its boundary S.

Let t denote the unit tangential vector of the boundary S such that

$$[n,t] = e_3, \tag{4.2}$$

e_3 denoting the unit vector in the x_3-direction. We decompose

$$\gamma = \eta t + \delta e_3 \quad \text{on } S, \tag{4.3}$$

this means

$$v = \eta e_3 - \delta t - \epsilon n \quad \text{on } S. \tag{4.4}$$

Then the integral equation (3.5) is split into the system of two equations

$$\tfrac{1}{2}\eta(x) + \int_S \frac{\partial\phi}{\partial n(y)}\, \eta(y)ds(y) - \lambda \int_S \Phi\delta(y)ds(y) \tag{4.5}$$

$$= (e_3, \text{rot} \int_B \Phi u(y)dy + \lambda \int_B \Phi u(y)dy),$$

$$\tfrac{1}{2}\delta(x) + \lambda \int_S (n(x),n(y))\Phi\eta(y)ds(y) - \int_S \frac{\partial\phi}{\partial n(x)}\, \delta(y)ds(y) \tag{4.6}$$

$$= \left(t, \text{grad} \left\{\int_B -\frac{1}{\lambda}\, \Phi \operatorname{div} u(y)dy + \int_S \Phi\epsilon(y)ds(y)\right\}\right)$$

$$-\left(t, \text{rot} \int_B \Phi u(y)dy + \lambda \int_B \Phi u(y)dy\right).$$

In the two-dimensional case the boundary value problem $K(B)$ can be reduced to solving a Dirichlet boundary value problem for the Helmholtz equation. By writing down the cartesian components of the equation $\mathrm{rot}\, v - \lambda v = u$ it is readily verified that the problem $K(B)$ is equivalent to solving the Dirichlet problem

$$\Delta v_3 + \lambda^2 v_3 = -(e_3, \mathrm{rot}\, u + \lambda u) \quad \text{in } B, \tag{4.7}$$

$$\frac{\partial v_3}{\partial t} = (n,u) - \lambda\epsilon \quad \text{on } S, \tag{4.8}$$

for the component v_3 in x_3-direction and then taking

$$\left.\begin{aligned} v_1 &:= \frac{1}{\lambda}\left(\frac{\partial v_3}{\partial x_2} - u_1\right) \\ v_2 &:= \frac{1}{\lambda}\left(-\frac{\partial v_3}{\partial x_1} - u_2\right) \end{aligned}\right\} \quad \text{in } B. \tag{4.9}$$

Because of the compatibility condition (3.3) we can choose a function $g \in C^{1,\alpha}(S)$ such that

$$\frac{\partial g}{\partial t} = (n,u) - \lambda\epsilon \quad \text{on } S, \tag{4.10}$$

and this function is uniquely determined up to an additive constant. Hence, we can uniquely determine g by the additional condition

$$\int_S g\, ds = 1. \tag{4.11}$$

Then the boundary condition (4.8) can be rewritten into the form

$$v_3 = g \quad \text{on } S. \tag{4.12}$$

Firstly, we consider the case where the homogeneous Dirichlet problem

$$\Delta a + \lambda^2 a = 0 \quad \text{in } B, \tag{4.13}$$

$$a = 0 \quad \text{on } S, \tag{4.14}$$

has only the trivial solution. Then the boundary value problem (4.7) and (4.8) is solvable and the solution can be made unique by prescribing

$$\int_S v_3\, ds = 1. \tag{4.15}$$

Since we have

$$\lambda(t,v) = -(t,u) - \frac{\partial v_3}{\partial n}$$

the solution of the system (4.5) and (4.6) is given by

$$\eta = g, \quad \delta = \frac{1}{\lambda}\left\{(t,u) + \frac{\partial v_3}{\partial n}\right\}. \tag{4.16}$$

Again, this solution can be made unique by the additional condition

$$\int_S \eta ds = 1. \tag{4.17}$$

Therefore, given $\eta = g$ explicitly by (4.10) to solve the boundary value problem we have to solve only one integral equation (4.6) for the unknown δ.

In the case where the homogeneous problem (4.13) and (4.14) possesses nontrivial solutions the inhomogeneous problem is solvable if and only if the additional condition

$$\int_S g \frac{\partial a}{\partial n} ds - \int_B (e_3, \mathrm{rot}\, u + \lambda u) a dx = 0 \tag{4.18}$$

is satisfied for all solutions a of (4.13) and (4.14).

Suppose the homogeneous problem (4.13) and (4.14) has exactly one linearly independent solution a and this solution satisfies

$$\int_B a dx = 0. \tag{4.19}$$

This, for instance, is true for B a rectangle. Then the problem

$$\mathrm{rot}\, v - \lambda v = \mathrm{grad}\, a \quad \text{in } B,$$

$$(n,v) = 0 \quad \text{on } S,$$

has a nontrivial solution v since (4.19) ensures that (3.3) is satisfied and (4.18) holds because from (4.10) we observe

$$\frac{\partial g}{\partial t} = \frac{\partial a}{\partial n} \quad \text{on } S$$

in this case, whence

$$\int_S g \frac{\partial a}{\partial n} ds = \frac{1}{2} \int_S \frac{\partial}{\partial t} g^2 ds = 0$$

follows. Therefore, in this particular case λ is not regular with respect to $K(B)$.

We emphasize at this point that for all λ the homogeneous problem $K(B)$ has a nontrivial solution. For those λ for which the homogeneous Dirichlet problem has only the trivial solution, there exists exactly one linearly independent solution of the homogeneous problem $K(B)$ and this solution is characterized by the property v_3 = const on S.

If the homogeneous Dirichlet problem (4.13) and (4.14) has m linearly independent solutions then obviously the homogeneous problem $K(B)$ has m linearly independent solutions with the property $v_3 = 0$ on S. In addition, there exists a further nontrivial solution with v_3 = const $\neq 0$ on S in the case where

$$\int_B a dx = 0$$

for all solutions to the homogeneous Dirichlet problem.

The infinite cylinder might be considered as the limiting case of a torus with cross section B. An extension of the results from the two-dimensional case to the case of a torus is in preparation.

In the case when B is the interior of the unit circle the solutions to the homogeneous problem $K(B)$ can be given explicitly ([4], p. 42, [13]) in polar cylinder coordinates (r,ϕ,z) by

$$v_r = 0 \quad v_\phi = cJ_1(\lambda r) \quad v_z = cJ_0(\lambda r) \tag{4.20}$$

where c = const and J_0 and J_1 denote the Bessel functions of order zero and one. With the exception of the zeros λ of J_0 we have v_z = const $\neq 0$ on the boundary S. For the zeros λ of J_0 there holds $\int_B v_z dx \neq 0$ and therefore no solution exists with v_z = const $\neq 0$ on S in these cases. Furthermore, for the zeros λ of the Bessel function J_m of order m we have the additional solutions

$$v_r = m \frac{J_m(\lambda r)}{\lambda r} (c_1 \cos m\phi - c_2 \sin m\phi),$$

$$v_\phi = -J_m'(\lambda r)(c_1 \sin m\phi + c_2 \cos m\phi), \tag{4.21}$$

$$v_z = J_m(\lambda r)(c_1 \sin m\phi + c_2 \cos m\phi),$$

with the property $v_z = 0$ on the boundary S. Note that all the streamlines are spirals.

5. Numerical solution of the integral equation in two dimensions

In order to get numerical approximations to the nontrivial solution of the homogeneous problem $K(B)$ after setting $\eta = 1$ we have to numerically solve the integral equation

$$\tfrac{1}{2}\delta(x) - \int_S \frac{\partial\Phi}{\partial n(x)}\,\delta(y)ds(y) = -\lambda \int_S (n(x),n(y))\Phi ds(y). \tag{5.1}$$

We assume the boundary S of B to be analytic and choose a parametric representation of the form

$$x(\sigma) = (x_1(\sigma),x_2(\sigma)), \quad 0 \leqslant \sigma \leqslant 2\pi.$$

Then, by straightforward calculations, we transform the integral equation (5.1) into the parametric form

$$\psi(\sigma) - \frac{1}{2\pi}\int_0^{2\pi} K(\sigma,\tau)\psi(\tau)d\tau = \frac{1}{2\pi}\int_0^{2\pi} L(\sigma,\tau)d\tau \tag{5.2}$$

where

$$\psi(\sigma) := ([\dot{x}_1(\sigma)]^2 + [\dot{x}_2(\sigma)]^2)^{\frac{1}{2}}\delta(x(\sigma)),$$

$$K(\sigma,\tau) := -i\pi\lambda\{\dot{x}_2(\sigma)[x_1(\sigma)-x_1(\tau)] - \dot{x}_1(\sigma)[x_2(\sigma)-x_2(\tau)]\}\,\frac{H_1^1(\lambda r(\sigma,\tau))}{r(\sigma,\tau)},$$

$$L(\sigma,\tau) := -i\pi\lambda\{\dot{x}_1(\sigma)\dot{x}_1(\tau) + \dot{x}_2(\sigma)\dot{x}_2(\tau)\}H_0^1(\lambda r(\sigma,\tau)),$$

$$r(\sigma,\tau) := \{[x_1(\sigma)-x_1(\tau)]^2 + [x_2(\sigma)-x_2(\tau)]^2\}^{\frac{1}{2}}.$$

Here $H_1^1(\xi) = -(d/d\xi)H_0^1(\xi)$ denotes the Hankel function of the first kind and of order one. By decomposing

$$H_0(\xi) = J_0(\xi) + iN_0(\xi)$$

where J_0 and N_0 denote the Bessel and Neumann functions of order zero and taking into account the expansions

$$J_0(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k\xi^{2k}}{(k!)^2 2^{2k}},$$

$$N_0(\xi) = \frac{2}{\pi}\left(\ln\frac{\xi}{2} + c\right) J_0(\xi) + \sum_{k=1}^{\infty} a_k\xi^{2k}$$

with real coefficients a_k and Euler's constant $c = 0.57721\ldots$ we observe that the integrands in (5.2) have logarithmic singularities. Therefore, we split

$$K(\sigma,\tau) = K_1(\sigma,\tau)\ln 4\sin^2\frac{\sigma-\tau}{2} + K_2(\sigma,\tau),$$

$$L(\sigma,\tau) = L_1(\sigma,\tau)\ln 4 \sin^2 \frac{\sigma-\tau}{2} + L_2(\sigma,\tau),$$

where

$$K_1(\sigma,\tau) := \lambda\{\dot{x}_2(\sigma)[x_1(\sigma) - x_1(\tau)] - \dot{x}_1(\sigma)[x_2(\sigma) - x_2(\tau)]\frac{J_1(\lambda r(\sigma,\tau))}{r(\sigma,\tau)},$$

$$K_2(\sigma,\tau) := K(\sigma,\tau) - K_1(\sigma,\tau)\ln 4 \sin^2 \frac{\sigma-\tau}{2},$$

$$L_1(\sigma,\tau) := \lambda\{\dot{x}_1(\sigma)\dot{x}_1(\tau) + \dot{x}_2(\sigma)\dot{x}_2(\tau)\}J_0(\lambda r(\sigma,\tau)),$$

$$L_2(\sigma,\tau) := L(\sigma,\tau) - L_1(\sigma,\tau)\ln 4 \sin^2 \frac{\sigma-\tau}{2}.$$

Then K_1, K_2, L_1, L_2 are analytic for all $0 \leqslant \sigma,\tau \leqslant 2\pi$. In particular

$$K_2(\sigma,\sigma) = \frac{\dot{x}_2(\sigma)\ddot{x}_1(\sigma) - \dot{x}_1(\sigma)\ddot{x}_2(\sigma)}{[\dot{x}_1(\sigma)]^2 + [\dot{x}_2(\sigma)]^2},$$

$$L_2(\sigma,\sigma) = \lambda\{[\dot{x}_1(\sigma)]^2 + [\dot{x}_2(\sigma)]^2\}\{2c + ln\tfrac{1}{4}\lambda^2\{[\dot{x}_1(\sigma)]^2 + [\dot{x}_2(\sigma)]^2\} - \pi i\}.$$

For the numerical approximation we choose an equidistant set of knots

$$\sigma_k := \frac{\pi}{N}k, \quad k = 0, \ldots, 2N-1,$$

and use the quadrature formulae

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)d\sigma \approx \frac{1}{2N}\sum_{k=0}^{2N-1} f(\sigma_k), \tag{5.3}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\ln 4 \sin^2 \frac{\sigma}{2}\, d\sigma \approx \sum_{k=0}^{2N-1} R_k f(\sigma_k), \tag{5.4}$$

where the weights R_k are given by

$$R_k := -\frac{1}{N}\left\{\frac{(-1)^k}{2N} + \sum_{j=1}^{N-1}\frac{\cos j\sigma_k}{j}\right\}, \quad k = 0, \ldots, 2N-1.$$

These quadrature rules are obtained by replacing f by its trigonometric interpolation polynomial and then integrating analytically. The quadrature rule (5.4) was previously used by Martensen [15] and Kussmaul [11] for the numerical solution of boundary integral equations with logarithmic singularities. Provided f is analytic, according to derivative-free error estimates in the spirit of Davis' method [2] for the remainder term in trigonometric interpolation of periodic analytic functions [7], the error of the quadrature rules (5.3) and (5.4) decreases at least exponentially when the number N of knots is increased.

In the well known fashion the integral equation (5.2) is now replaced by the approximate linear system

$$\psi_k - \sum_{j=0}^{2N-1} \left\{ R_{j-k} K_1(\sigma_k, \sigma_j) + \frac{1}{2N} K_2(\sigma_k, \sigma_j) \right\} \psi_j$$

$$= \sum_{j=0}^{2N-1}{}' \left\{ R_{j-k} L_1(\sigma_k, \sigma_j) + \frac{1}{2N} L_2(\sigma_k, \sigma_j) \right\}, \quad k = 0, \ldots, 2N-1.$$

The numerical example is carried out for an ellipse

$$x_1(\sigma) = \cos\sigma, \quad x_2(\sigma) = b \sin\sigma.$$

From (4.20) we observe, that in the limiting case $b = 1$ of the unit circle, the solution of the integral equation (5.1) is given by $-\delta = J_1(\lambda)/J_0(\lambda)$.

We conclude with a few numerical results on the values of $-\delta$ on the boundary for $b = 1$, 0.6, 0.4 and $N = 16,8$.

$\lambda = 0.1 \quad N = 16$

σ \ b	1	0.6	0.4
0	0.050062	0.044140	0.034489
$\pi/8$		0.044142	0.034490
$\pi/4$		0.044146	0.034493
$3\pi/8$		0.044149	0.034495
$\pi/2$		0.044150	0.034497

$\lambda = 0.1 \quad N = 8$

σ \ b	1	0.6	0.4
0	0.050062	0.044149	0.034911
$\pi/4$		0.044144	0.034438
$\pi/2$		0.044146	0.034366

$\lambda = 0.5 \quad N = 16$

σ \ b	1	0.6	0.4
0	0.258152	0.223502	0.173247
$\pi/8$		0.223695	0.173390
$\pi/4$		0.224143	0.173737
$3\pi/8$		0.224632	0.174087
$\pi/2$		0.224827	0.174232

$\lambda = 0.5 \quad N = 8$

σ \ b	1	0.6	0.4
0	0.258152	0.223555	0.175428
$\pi/4$		0.224156	0.173449
$\pi/2$		0.224807	0.173570

$\lambda = 1 \quad N = 16$

σ \ b	1	0.6	0.4
0	0.575080	0.466078	0.351683
$\pi/8$		0.467751	0.352873
$\pi/4$		0.471823	0.355777
$3\pi/8$		0.475938	0.358717
$\pi/2$		0.477655	0.359945

$\lambda = 1 \quad N = 8$

σ \ b	1	0.6	0.4
0	0.575081	0.466361	0.356544
$\pi/4$		0.471676	0.355269
$\pi/2$		0.477730	0.358643

The numerical results indicate that the field lines which are spirals ([4], p. 42) become steeper as the minor axis b gets smaller and that they are slightly steeper at the point $\sigma = 0$ (major axis) than at the point $\sigma = \pi/2$ (minor axis).

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